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Student Research Working Paper Series

Portfolio Construction and Optimization

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PORTFOLIO CONSTRUCTION

AND OPTIMIZATION

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1. INTRODUCTION

One can rarely find an investor these days holding all of his/her assets in a isolation. From small to big, private to institutional investors as well as asset management firms everyone spread their wealth across multiple assets even multiple asset classes including equity, fixed income, foreign exchange, derivative products, etc. In other words, every financial market participant is likely to hold a portfolio. Therefore, portfolio dynamics and benefits are more relevant than ever.

1.1 Practical objective

Undergraduate finance majors are likely to get exposure to Modern Portfolio Theory (MPT) developed by Harry Markowitz[1] (1952). Classical MPT model introduces the concepts of diversification through combining imperfectly correlated assets as well as efficient frontiers. Standard portfolio courses build on MPT with later developments on the subject mostly from William Sharpe[2] (1964) introducing Capital Asset Pricing Model (CAPM) and risk-reward measure – Sharpe Ratio, maximization of which is often used as objective function in portfolio optimization.

Introductory portfolio courses serve their purpose by providing fundamental understanding on how investment portfolios are constructed. Unfortunately, such introduction is highly insufficient in any practical sense. Without further interest into the subject is essentially inapplicable in practice for two main reasons - oversimplification of exercise scenarios as well as invalidity classical model assumptions.
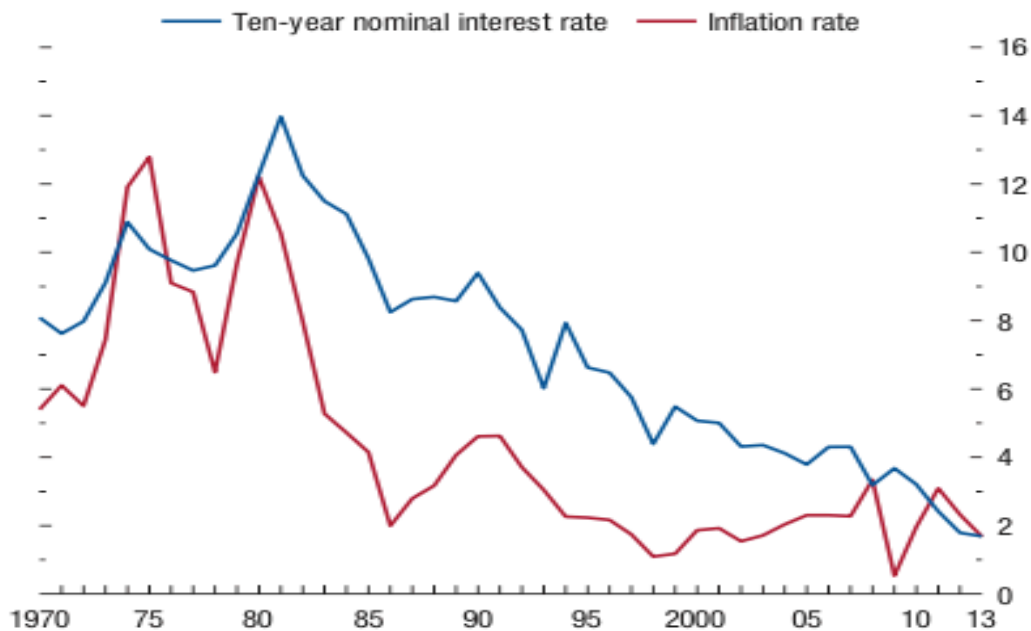
Figure 1: Government Bond and Inflation Rates

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[1] Markowitz, H.M. (March 1952). Portfolio Selection. The Journal of Finance 7 (1), 77-91
[2] Sharpe, William F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk, Journal of Finance, 19 (3), 425–442

Sources: Bloomberg, L.P.; Haver Analytics; Organization for Economic Cooperation and Development; World Bank, World Development Indicators database; and IMF staff calculations.
Note: Inflation is calculated as the percent changes in the consumer price index.

Investment scenarios supplied by introductory textbooks are often based on too few constituents (frequently as little as two assets) and long only composition. While diversification gains can be successfully demonstrated with few assets, more optimal idiosyncratic risk reduction occurs when 50 or more assets are held in a portfolio. Short-selling is another important component of a well diversified portfolio that can be used for both bearish bets as well as hedging or increasing market neutrality. This is particularly important for hedge funds.

Also we have to into account unrealistic MPT assumptions. Among the major model flaws are assumed normality, volatility and correlation stability over time, absence

of real performance measuring (accounting for diversification and optimization costs), infinite liquidity and unrestricted divisibility of assets, etc.

Practical objective of this paper will be to address some of the outlined Modern Portfolio Theory invalid assumptions that are within reasonable complexity as well as conduct empirical analysis and portfolio construction on a larger, more realistic scale.

1.2 Market objective

Motivation for this paper is also greatly impacted by current market environment. As outlined by International Monetary Fund World Economic Outlook, global interest rates and inflation was gradually decreasing since 1980's (see Figure 1). However, over the last few years rates have dropped to critical levels. Earlier this year interest rates became negative in several markets including Germany (get source). This zero bound phenomenon beginning to result in a investment paradigm shift. With interest rates underperforming bonds are becoming absolete. Fixed rate securities historically served as a safety investment to balance out equity market fluctuations at the same time providing moderate and stable returns. However, bonds no longer can provide its usual function in the current environment. Therefore, there is a need among various investors to rethink their long-term asset allocation strategies. Without getting into complex structured products in order to engineer fixed cash-flow securities yielding higher rates, potential solution might be in efficient equity portfolio optimization with return stability and minimal risk in mind.

For this reason, alongside portfolio construction and MPT model extension, the secondary focus of this paper will be on portfolio optimization. I will conduct a survey of

various portfolio optimization methods and explore their behavior over time and different market environments.

Section 2 of this paper describes data and main variables used in this paper. Section 3 deals with risk and return modeling. Portfolio construction and optimization is done in Section 4. Finally, Section 5 presents concluding statements.

2. DATA AND VARIABLES

Market data used in this paper consists of daily closing prices of 502 US equities from the current S&P 500 (SPX) Large-Cap Index universe. The data set has 2,516 observations (trading days) covering the 10-year period from 09-01-2005 to 08-31-2015. Daily prices frequency ranges from 450 to 502 available stocks. This period was selected to reflect full economic cycle representing second half of the US mortgage boom leading into financial crisis of 2007-2008, followed by recession and recovery periods. Historical prices are obtained from Bloomberg Terminal.

Daily logarithmic returns are computed from daily prices as:

$$r_t = \ln(\frac{P_t}{P_{t-1}})$$

Annualized volatility is calculated from 30-day (N) trailing returns assuming 252 trading days as:

$$vol = \sqrt{\frac{252}{N} \sum_{t=1}^{N} r_t^2}$$

Custom S&P 500 Index will be used to benchmark optimized portfolio performance. Daily index returns are computed as equal-weighted average of daily

constituent returns. Note: this index does not correspond to real S&P 500 Index as members of an actual index are not constant over time.

3. MODELLING

3.1 Return distribution

Before we begin any portfolio modeling it is necessary to revisit the fundamental flaws of MPT stated in the introduction. Perhaps the biggest one is assumed normality. According Markowitz's model, asset returns are supposed to follow normal (Gaussian) distribution. However, that is far from reality. Various empirical studies on the behavior of financial asset return time series collectively described as stylized facts[3] refute normality by indicating pronounced higher moments as well as suggest returns being not independently and identically distributed random variables (i.i.d.).

Financial return distributions can be characterized with fat tails and asymmetry. It is more likely for abnormal profits or losses to occur in financial time series than normal distribution implies. Also returns tend to be slightly left skewed suggesting higher probability of loss compared to gain. I.i.d. is rejected due to phenomenon know as volatility clustering meaning that volatility spikes (extreme returns) occur in bursts. See Figures 3.1 and 3.2 for demonstration of stylized facts for custom S&P 500 Index.

Figure 3.1 compares empirical index return distribution to fitted normal distribution. Visually it is clear that custom index returns are non-normally distributed. Empirical returns cluster tighter around the mean, but have noticeable more pronounced tails. Figure 3.2 illustrates higher-level analysis of heavy tails. Vertical bars in this plot represent daily return moves of ± 2.326 standard deviations and higher. Assuming

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[3] Cont, R. (2001). "Empirical properties of asset returns: stylized facts and statistical issues". Quantitative Finance. Volume 1 p.223-226

normality of returns, such events no higher than 1% chance of occurring on either tail. There are 29 positive and 69 negative extreme return moves empirically representing ~1.17% and ~2.78% probability respectively. On either end probabilities are above their normal likelihood, with negative side being significantly higher that expected. This is once again in line with stylized characteristics.

Figure 3.1: S&P 500 return distribution
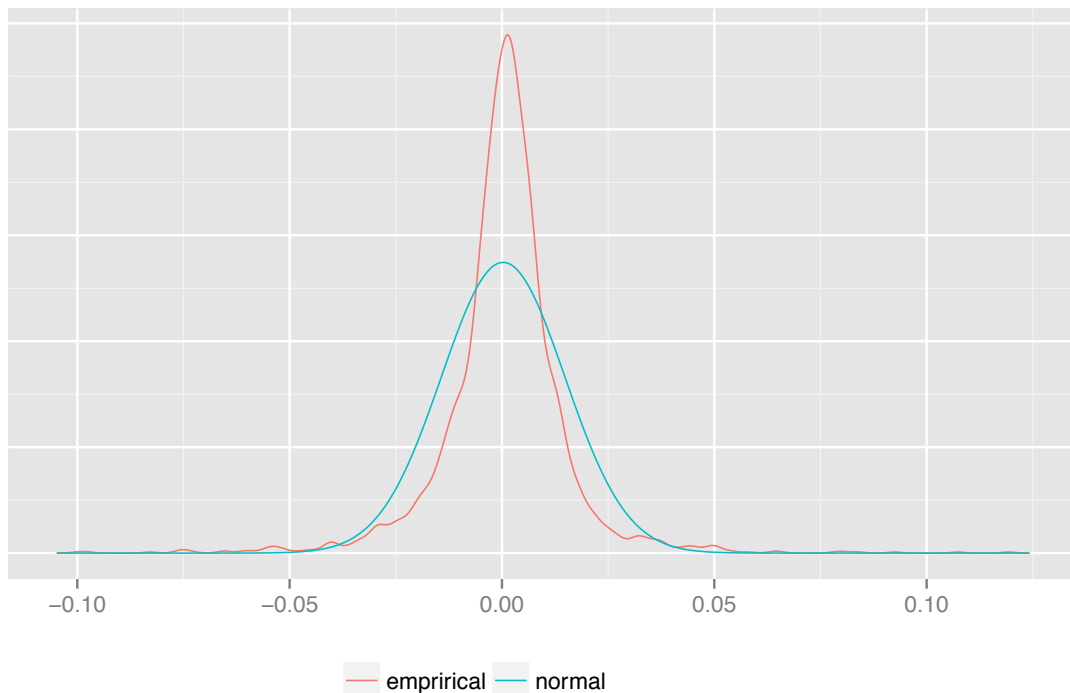
The time series of returns is a foundation on which portfolio and risk models are built. Therefore, it is extremely important to recognize these errors and correct for them. If we were to build an investment portfolio based on return normality and then apply Gaussian risk models, we would put ourselves at great risk potentially underestimating frequency and magnitude of extreme events. And as previously noted most likely not the positive surprises.

Figure 3.2: S&P 500 extreme returns
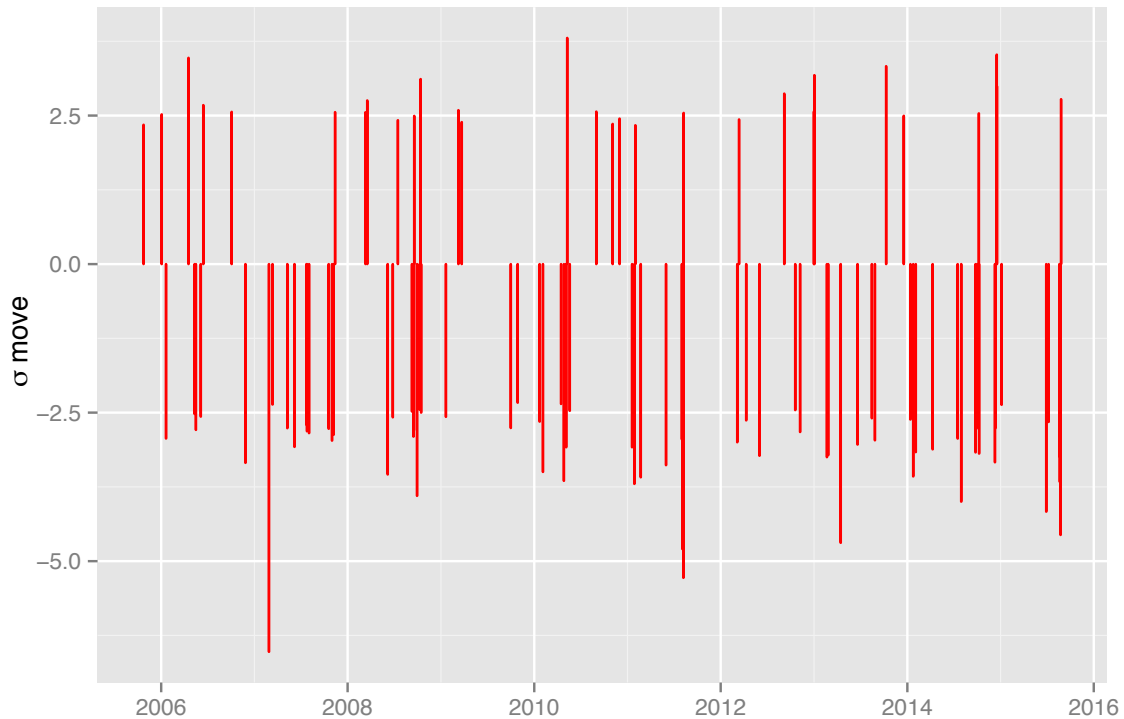
Finally, given misleading risk measures optimization itself loses efficiency and credibility, as risk is part of virtually every optimal portfolio consideration. Fortunately, there are other distributions at our disposal.

Given the stylized asset return time series characteristics, the optimal distribution to model equity returns should be flexible in terms of higher moments. The continuous density function should have skewness and kurtosis parameters. The list of potential candidate distributions is too long to list. However, having risk modeling as an end goal it is possible to focus on the most prominent ones.

Financial literature on financial risk modeling suggests Generalized Hyperbolic Distribution (GHD) family as a superior alternative to normal distribution. Hu and

Kercheval (2007) argue that GHD and certain special cases of it produce more accurate asset return density model.[4]

The GHD is five parameter ($\lambda, \alpha, \beta, \delta, \mu$) based density generating function. By manipulating these parameters responsible for class definition, shape, scale and location we can generate a spectrum of distributions capturing heavier tails, hence GDH being a family of distributions. Several special cases include:

- Student's t-distribution, with $\lambda = -v/2$, α and $\beta = 0$, $\delta = v^{1/2}$ (v – degrees of freedom)

- Hyperbolic distribution (HYP), with $\lambda = 1$

- Normal-inverse Gaussian (NIG), with $\lambda = -1/2$

- Variance-gamma distribution, with $\delta = 0$

Density function, parameter scope, and parametrizations are defined as follows[5]:

$$gh(x; \lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta)(\delta^2 + [\![(x-\mu)^2]\!]\,{}^{((\lambda-1/2)/2)} \times$$

$$K_{(\lambda-1/2)}(\alpha\sqrt{(\delta^2 + (x-\mu)^2})})\exp(\beta(x-\mu)$$

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\,\alpha^{-\frac{\lambda}{2}}\,\delta^\lambda\,K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}$$

,where x is a random variable. Parameter range is defined as $\lambda, \mu \in R, \delta > 0$ and $0 \leq |\beta| < \alpha$. K_λ function and additional parameters ζ and ξ are defined as well as some special case distributions derived by Prause (1997). Due to complex multi-parameter approximation and calibration, data fitting and parametrization is done with 'ghyp' library for R

[4] Hu, Wenbo., Alec Kercheval. (2007). Risk management with generalized hyperbolic distributions. FEA 2007 Proceedings of the Fourth IASTED International Conference on Financial Engineering and Applications. p 19-24

[5] Prause, Karsten. (September 1997). Modeling Financial Data Using Generalized Hyperbolic Distributions. Freiburg Center For Data Analysis and Modeling
<http://www.dms.umontreal.ca/~morales/docs/prause_ghe.pdf>

programming language.

Table 3.1 records summary of GHD fitting parameters and quality results. It appears in this scenario asymmetric normal inverse Gaussian distribution produces slightly better fit than other models as determined by highest Akaike Information Criterion (AIC) and log-likelihood measures. Other five parameters dented by Greek letters represent shape – λ and α-bar, location – μ, dispersion – σ, and skewness – γ.

Table 3.1							
Model	λ	α bar	μ	σ	γ	AIC	Log-likelihood
NIG	-0.500	0.283	0.001	0.014	-0.001	-15113	7561
GHD	-0.521	0.280	0.001	0.014	-0.001	-15111	7561
t-dist	-1.132	0.000	0.001	0.022	-0.002	-15095	7551
VG	0.771	0.000	0.002	0.013	-0.001	-15055	7531
HYP	1.000	0.000	0.002	0.013	-0.001	-15032	7520

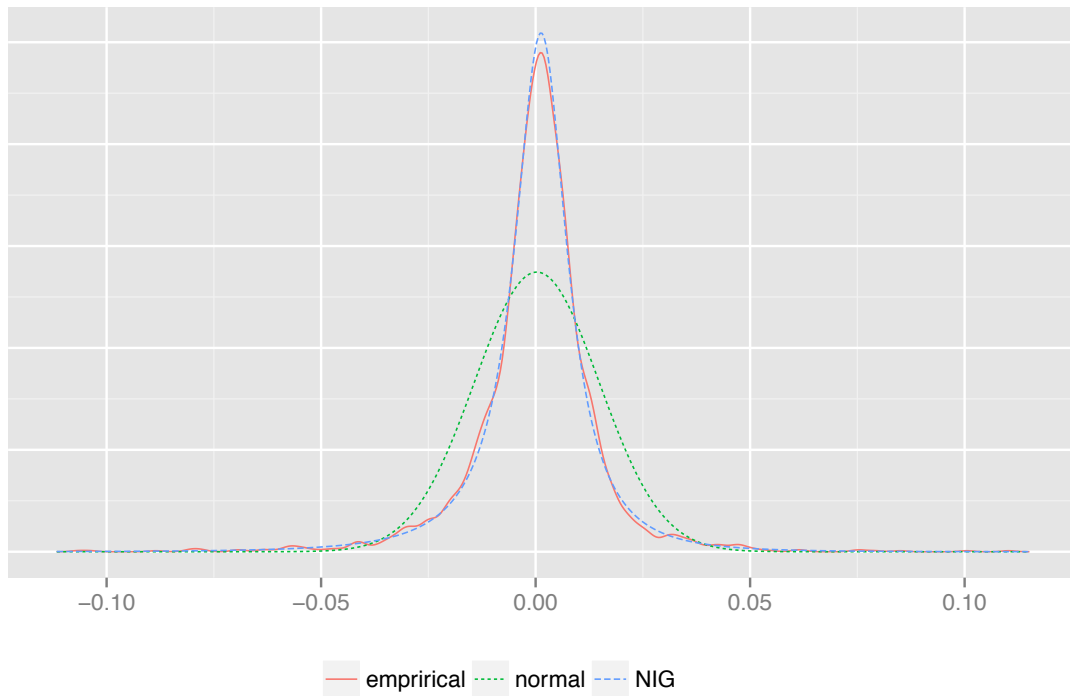## Figure 3.3: S&P 500 GHD return distributions

Figure 3.3 displays visual alignment of empirical, normal, and normal inverse Gaussian distributions. Density plot shows fairly similar curvature between empirical and NIG lines. Also referring to Figure 3.4 we can see value at risk (VaR) curve comparison for previously state distributions. NIG appears to produce much fatter tails assigning higher likelihood to far field events appropriate to real world asset return movement. This gives confidence to proceed with further risk and portfolio modeling having a more accurate representation of return processes than relying on the assumption of normality. Density plot overlaying all model curves is included in the Appendix section.

Figure 3.4: S&P 500 GHD VaR
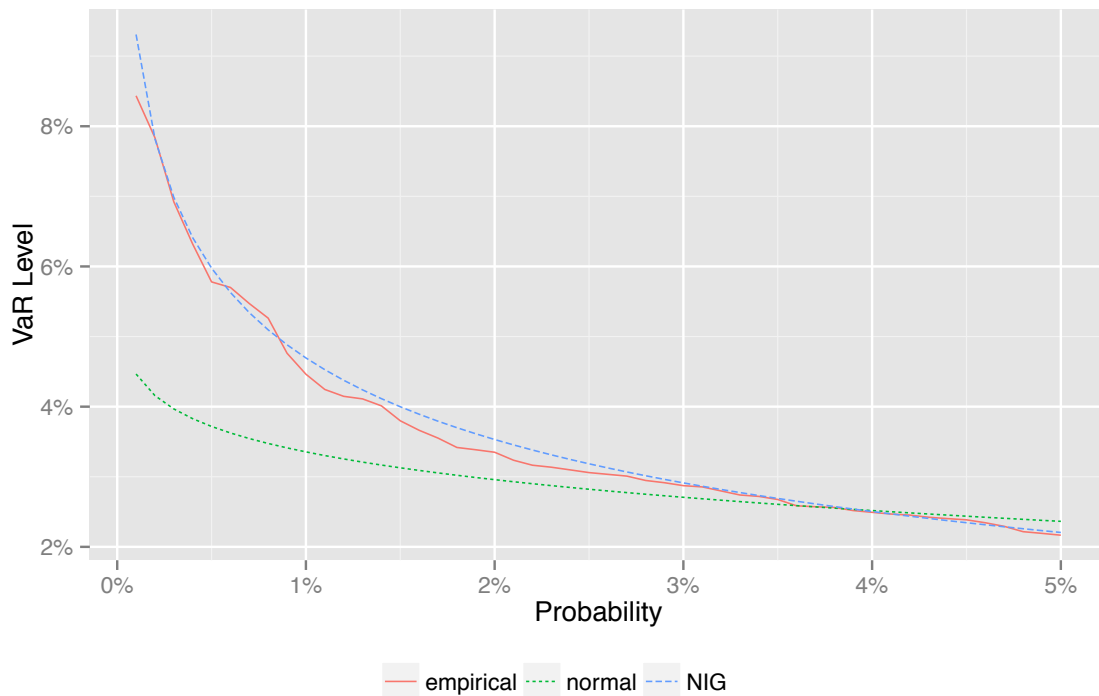
4. PORTFOLIO

This section will go through long-short portfolio construction, optimization techniques, and risk measures. Unfortunately, due to computational limitations only more

basic numeric examples will be provided. More complex optimization techniques will be conducted on a small subset of S&P 500 Index constituents.

First, we begin with the Modern Portfolio Theory cornerstone that is empirical exploration of gains from diversification of a large-scale portfolio. Equal-weighted standard deviation of 503 individual stocks over the entire observed time period amounts to 22.41%. Same stocks also held in equal share in a portfolio exhibit overall 14.54% standard deviation covering the same period of time. That is a realized 35% reduction in risk.

Building on the work of Markowitz, the next step is to compute global minimum variance (GMW) portfolio and evaluate return and volatility performance. Standard deviation of this long/short GMV portfolio was realized at 4.70% or another 68% decrease in overall risk and 79% total risk reduction from isolated individual index members. This dramatic risk reduction was achieved without any sacrifices in relative cumulative performance. Figure 4.1 shows wealth trend of $1 invested in both equal-weighted S&P 500 Index and GMV Portfolio. This net long portfolio with roughly equal number of long and short position performed exceptionally well during crisis period when it suffered only 4.47% theoretical loss compared to 29.23% loss for S&P 500 index. Short hedge implementation increasing portfolio's market neutrality speaks for itself.

The following Figure 4.2 visualizes annual volatility of long/short GMV relative to S&P 500 Index. The graph is completely red. The optimized portfolio outperforms non-optimized portfolio in terms return movement 100% of the time. This short-side

enabled introductory level optimization problem applied to sufficiently large-scale data strongly promotes benefits of statistical analysis for performance gains.

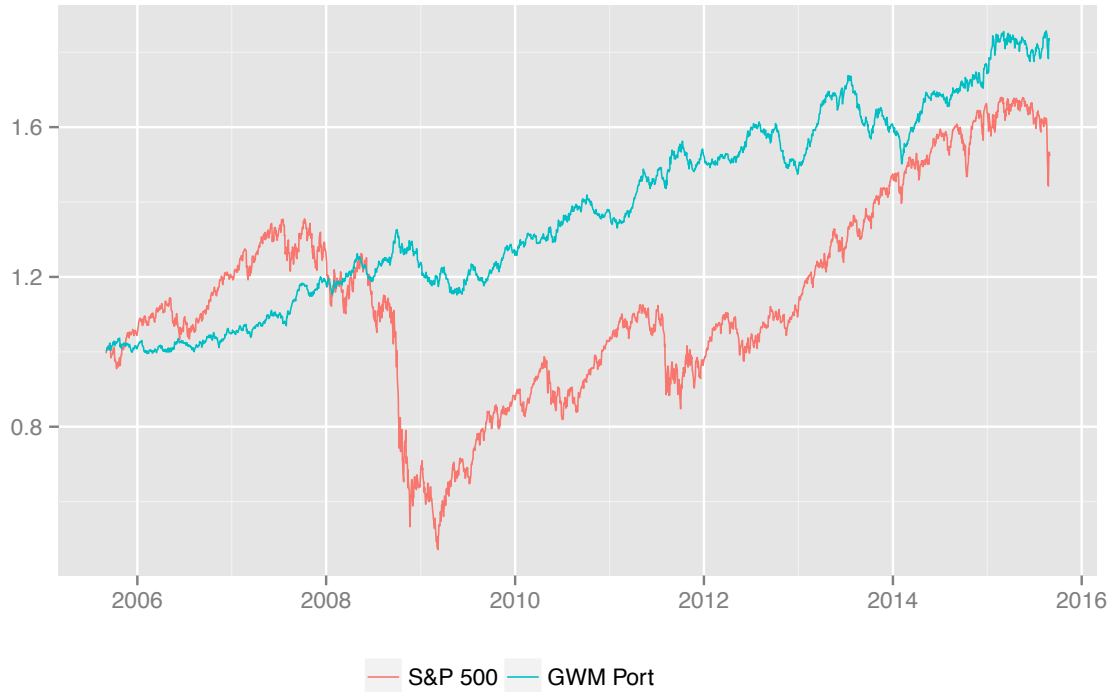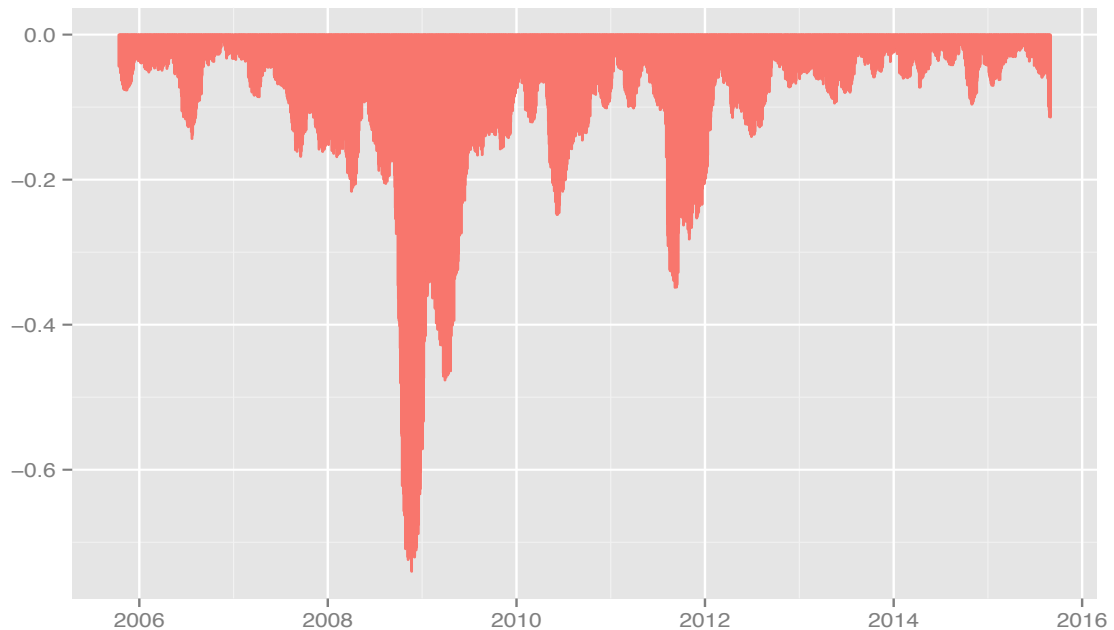## Figure 4.1: S&P 500 and GMW port performance

Figure 4.2: GMW Port vs. S&P 500 Index Volatility

It is perhaps a good time to talk about risk. Just a moment ago standard deviation was used to as a mean to measure risk of two portfolios as well as individual stocks. However, in practice standard deviation is rarely acceptable measure of risk. Earlier in this paper value at risk levels (VaR) were used to compare potential magnitude of a loss given a certain probability for different distributions. VaR is the risk measure of choice when it comes to teaching topics related to financial risks. Nevertheless, since 1999 after Philippe Artzner et. al. published a paper titled Coherent Measures of Risk[6], VaR was being criticized for not being "coherent." In this case coherence was defined by four desirable parameters – monotonicity, sub-additivity, homogeneity, and translation invariance. Out of four prerequisites VaR fails to satisfy sub-additivity.

However, upon declaring VaR incoherent Artzner provided a solution. He introduced a risk measure originally called Expected Shortfall (ES), but is more widely known these days as Conditional Value at Risk (CVaR). CVaR doesn't only return minimum expected loss for a given alpha. It takes into account the entire tail and therefore, returns average VaR from 1-α to 1. ES is defined as follows:

$$ES_\alpha = -\frac{1}{\alpha}\left(E\left[X\,1_{\{X \leq x_\alpha\}}\right] + x_\alpha(\alpha - P[X \leq x_\alpha])\right)$$

We can now apply CVaR to the two generated portfolios and model their risk. In the previous section NIG was determined produce the best fit to model asset returns. We will be using NIG density function to fit return profile of long/short GMV portfolio and compare its CVaR trend line with that of S&P 500 displayed in Figure 4.3. There seems to be some unexpected tail behavior related to S&P 500. CVaR curves for both NIG distributed and empirical returns (not in graph) exhibit distorted risk pattern that requires

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[6] Artzner, Philippe et. al. (1999). Coherent Measures of Risk. Mathematical Finance 9 (3): 203

further investigation. On the other hand GMV portfolio shows impressive tail risk performance.

Figure 4.3: CVaR with NIG distribution
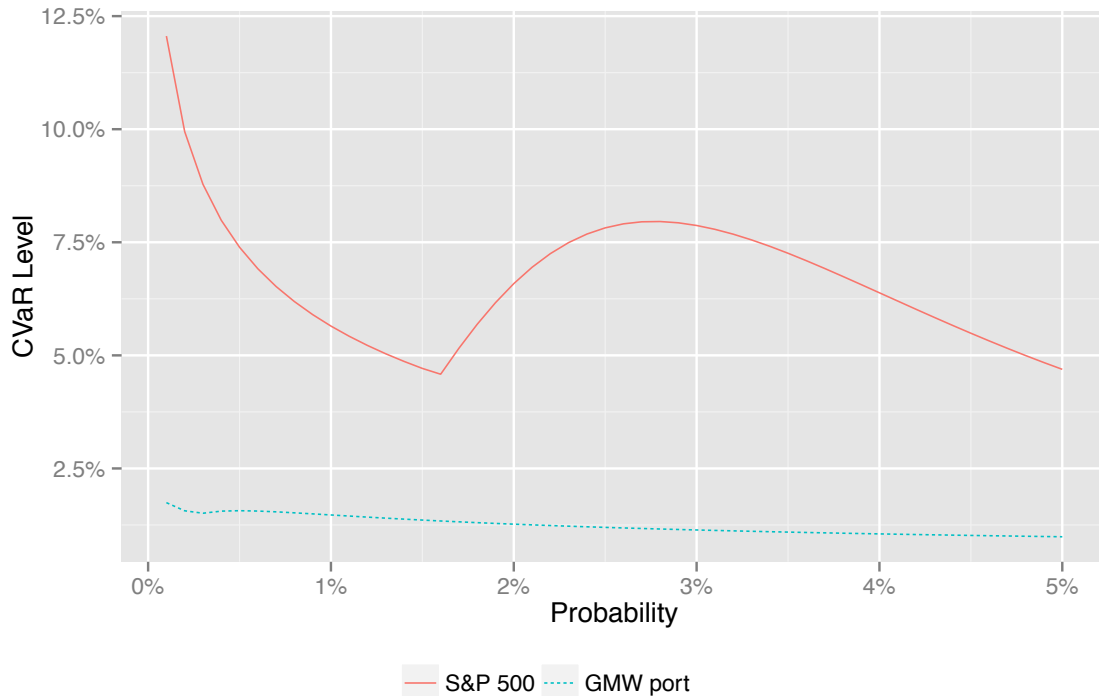
As previously stated in the introduction, one of the objectives of this paper is to investigate whether or a particular optimization technique can produce equity return stream, similar to that of pre-zero-bound fixed income cash flows. The objective here is clear – generate moderate return levels with risk levels close to that of investment grade corporate bonds to replace low yielding current fixed income securities. This little GMV experiment produced a good starting and benchmarking point for further modeling. Ideally a optimization technique producing a portfolio with less than 3% standard deviation and annualized total returns in the 5-7% range.

Naturally an initial hypothesis at this point could be that there may not be a better optimization model to reduce risk than GMV as it already implies potentially minimum

variance portfolio solution. Especially considering very solid theoretical performance. Note: GMV model produced in the previous example is not the most robust implementation. It does violate several of the stylized facts of asset return time series the most important one being assumption of constant correlation.

Current model generates variance covariance matrix based on the full return spectrum returning portfolio asset co-dependencies as of last observation date that remain constant going backwards in time. However, asset volatilities vary over-time in turn changing correlations in the system.

In order to address this violation, the use of time-varying volatility based models is advocated. According to literature, the best model to deal with time series of variance is a generalized autoregressive conditional heteroskedasticity[7] (GARCH) model. The model is based on previously mentioned volatility clustering present in financial markets by which it returns conditional volatility with respect to time and event models it into the future. Therefore it is certainly outlining this potential upgrade to any portfolio optimization problem. However, due to its complexity it won't be implemented in this paper.

Going back to the search of potential improvement over GMV portfolio optimization it appears a challenging task. Long/short GMV set a very high benchmark. Despite that several potentially competing optimization objectives come in mind. As CVaR measure was introduced with focus on the extreme event modeling, it does make sense to try to optimize for it. Maybe the solution is not the minimum overall variance, but rather left tail minimization. Another higher level variation of it could be maximizing

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[7] Bollerslev, Tim. (1986). Generalized Autoregressive Conditional Heteroskedasticity. Journal of Econometrics 31 p307-327

left to right tail CVaR (minimum loss, maximum profit expectation). Finally, we can look into drawdown optimization. This in theory should behave similarly as GMV, since we are looking for flatter return structure.

As these more advanced probabilistic optimizing objectives are very computationally intensive, only portfolios containing about 10 stocks can be optimized on my system given the number of observations. So I have selected a random sample of 10 stocks to form a test portfolio. Among the optimizations tested on the portfolio were GMV (for aligning purposes), long only GMV, and min CVaR long only. Figure 4.4 displays the performance comparison between highest and lowest performing optimum portfolios.

Figure 4.4: GMW and Min CVaR port performance

— GWM 10 Stock Long/Short — Min CVaR 10 Stocks Long
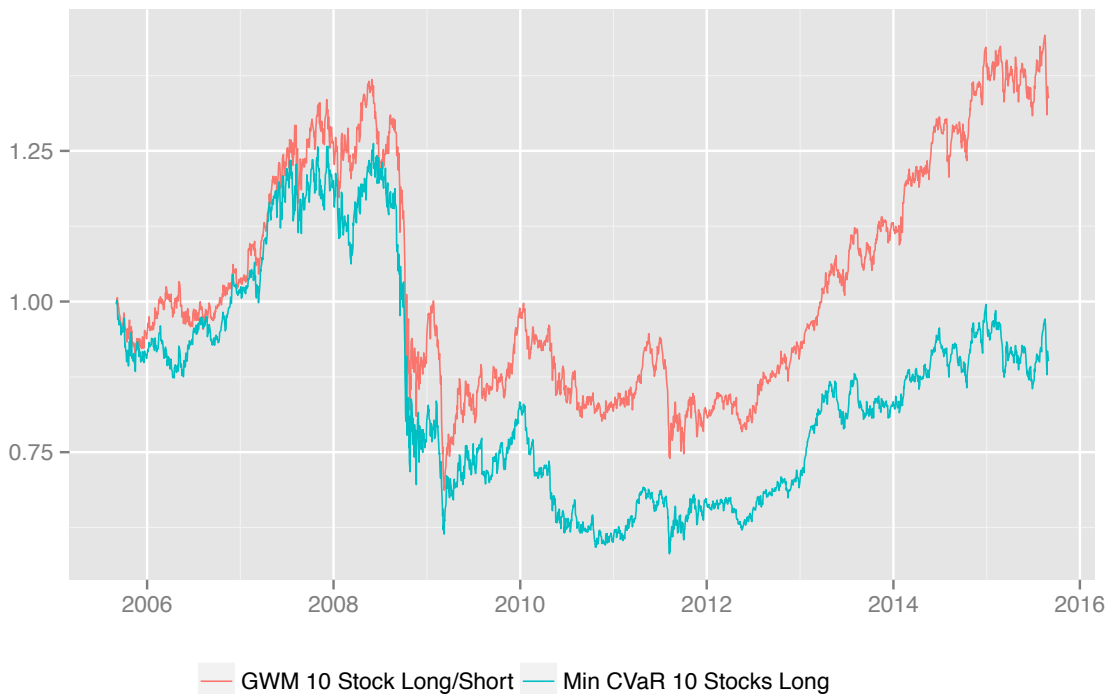
Min CVaR portfolio is clear underdog in this case, while long/short GMV keeps the lead. Although, due to a low number of constituents selected insufficient

diversification is also apparent. GMV lost its flatter return pattern. In terms of volatility of these optimal portfolios, long/short GMV has then lowest standard deviation of 12.1% which is still lower that a full 500 stock equal-weighted index. Long only GMV (not displayed) is at slightly higher st.dev of 12.4% and lower cumulative return proving performance gains by shot hedging. Finally, with significantly lower return profile min CVaR portfolio also has a higher standard deviation of 13.5%.

5. CONCLUSION

The search for bond-like equity portfolio unfortunately due to computational limitations did not yield very surprising results. Long/short GMV produced remained undefeated by both return and volatility patterns. Perhaps the final solution is to focus on robustness of GMV estimators (such as GARCH modeling), play around with additional constraints, introduce portfolio rebalancing, and even try out certain equity selection strategies rather than including the entire index universe.

S&P 500 GHD return distributions – all models